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DEVELOPMENT OF AN EDUCATIONAL PROGRAM FOR VISUALIZATION OF THE FOUR-COLOR THEOREM

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Abstract

The four-color theorem is one of the most interesting theorems in graph theory. The theorem is also interesting because, in addition to its rigorous mathematical formulation, it can also be easily illustrated on real maps, which makes it easier to understand the nature of the theorem. This is especially important in the educational field, where a visual or schematic explanation can significantly facilitate the understanding of complex theoretical material. Although the theorem was proposed more than 170 years ago and has been studied extensively, there are few software tools that can illustrate the application of the theorem to real maps. Some of the available tools work only with a map of a specific country, while others allow you to color the map yourself. The goal of this work is to design and develop a software tool that will display not only the result of map coloring, but also all the steps of the applied algorithms: the transition from a map to a planar graph, the enumeration of vertices, and the proper vertex coloring of the graph. The tool developed in the work allows to apply the four-color theorem to maps of different countries, as well as a comparison is made between some well-known coloring algorithms.

Keywords: planar graphs, four color theorem, vertex coloring of a graph, proper coloring of a map, greedy algorithm, Dsatur, backtracking.

Introduction

Graph theory is one of the modern branches of discrete mathematics, and planar graphs are one of the most important subfields of graph theory. One of the fundamental results of planar graph theory is the four-color theorem. On October 23, 1852, mathematician Francis Guthrie, while coloring a map of the counties of England, noticed that it was enough to use four colors to color the map [2]. The exact formulation of the theorem was given by Arthur Cayley (1879). The theorem states the following: *Any geographical map can be colored using four colors, so that countries with a common border (neighboring) are colored in different colors* (here we understand that the territory of each country consists of a single connected region and we consider two countries to be neighbors if they have a common border in the form of a line, not a point) [1]. Or, equivalently, *for any* ***planar graph*** *G (a*

graph that can be drawn without edge intersections or self-intersections), its chromatic number is less than or equal to 4, i.e. $\chi(G) \leq 4$.

The theorem was proved more than 100 years later, on June 21, 1976, by Kenneth Appel and Wolfgang Haken of the University of Illinois [3]. The theorem was proved using a computer and was one of the first major mathematical proofs performed on a computer.

Conflict Setting

Since a geographical map is easily reduced to a planar graph (each administrative-territorial region is assigned a vertex, and the vertices are connected by an edge, if these regions have a common border, in fact a dual graph is constructed), then the problem of visualizing the four-color theorem is reduced to the problem of performing the proper vertex coloring of a planar graph with 4 colors (when neighboring vertices have different colors). The main goal of the developed educational tool is to graphically demonstrate the application of the theorem on real maps. To achieve this, geographical maps taken from the GADM version 4.1 database [7] were used. The maps are presented in json format, according to different levels of division (level-0: the entire country as a single unit, level-1: the country divided into administrative-territorial regions, level-2: regions divided into communities, etc.). Maps of countries that have a level-1 division and the theorem test makes sense for the given map (the regions have a common border, not enclaves or archipelagos) were taken. The main steps of the program are as follows:

1. Transition from a map to a planar graph
2. Application of any coloring algorithm to perform a proper vertex coloring
3. Demonstration of the construction of the graph and the coloring of the regions step by step in the animation format.

To implement this, a graphical user interface (GUI) was created using the Python customtkinter library, which allows the user to select the map of the desired country from the given list of countries, specify whether to show the construction of the graph or only the coloring and get the result in the form of an interactive animation. The map of the selected country is loaded into the program from a json file and saved using tools of the geopandas library. A list is created in the program, where the administrative-territorial regions of the selected map are stored as vertices. Then, pairs of vertices that are adjacent are stored as edges. Note that since the maps are given as sequences of boundary coordinates, in some cases there is a small gap between administrative-territorial units, which prevents accurate verification of the adjacency of regions. To solve this problem, a safety buffer was added that extends the boundary line by a small amount. As a result of experiments, a buffer value of 0.00001 degrees (~1.11 meters) was chosen, which solves the problem but does not distort the accurate data (it does not connect areas that are slightly distant from each other but do not have a common border with an edge). The created graph is stored in a networkx library object. After performing the coloring, the result is displayed in the program in an animation format using the matplotlib library.

Experimental procedures

The main problem that the developed program must solve is the proper vertex coloring of a planar graph obtained from a map. One of the well-known algorithms for vertex coloring

of a graph is the greedy algorithm. The essence of the algorithm is as follows: *traverse the vertices of the graph, and at each step, color the given vertex with the smallest possible color, making sure that none of the neighboring vertices is already colored with the same color* [1]. According to the fundamental property of greedy coloring, for an arbitrary graph $G = (V, E)$ the inequality $\chi(G) \leq d + 1$ holds, where d is the largest of the degrees of the vertices of the graph, $d = \Delta(G)$ [1].

It is known that in the case of greedy coloring, the ordering of the vertices is vital, since different orderings result in different colorings and the number of colors used may differ. That is, greedy coloring does not always give the best result, and therefore, the greedy coloring problem is reduced to the problem of ordering the vertices of a graph. One of the ordering options is LDF (Largest Degree First) [6]. In this algorithm, the vertices of the graph are first ordered in decreasing order of vertex degree, and then the greedy coloring algorithm is applied.

Although this and other coloring algorithms discussed later are available in the networkx library as functions, they were not used in our program and the coloring functions were implemented by ourselves, since in addition to coloring, the sequence in which the algorithm steps were performed is also necessary to create a coloring animation. By implementing coloring with this algorithm and applying it to the map of the Republic of Armenia, the following result was obtained (Fig. 1):



Fig. 1. Result of greedy coloring with LDF vertex ordering for the map of the Republic of Armenia

Applying the algorithm to maps of different countries shows that this algorithm does not always provide an optimal coloring, and in some cases more than four colors are used. For example, when coloring the map of the Federal Republic of Germany with this algorithm, the following result was obtained (Fig. 2):



Fig. 2 Result of greedy coloring with LDF vertex ordering for the map of the Federal Republic of Germany

The result shows that in this case 5 colors were used. A similar problem also arises when coloring maps of some other

countries, for example, the map of the Russian Federation. Therefore, this algorithm cannot be used to demonstrate the 4-color theorem.

To avoid this problem, let us consider another coloring algorithm, the Dsatur (Degree of Saturation) algorithm, which was proposed in 1979 by Daniel Brelas [4]. Before describing the steps of the algorithm, let us introduce the concept of vertex saturation.

*The **saturation degree** of a vertex is the number of different colors with which the neighboring vertices of the considered vertex are colored*.

The main steps of the algorithm are as follows:

1. Order the vertices of the graph in decreasing order of vertex degree
2. Color the first vertex with color 1
3. Select the vertex with the highest saturation degree from the uncolored vertices (if there are several vertices with the same saturation, then select the vertex with the most uncolored neighbors)
4. Color the selected vertex with the color of the smallest possible number
5. If there are still uncolored vertices, proceed to step 3, otherwise finish coloring.

By implementing coloring with this algorithm and applying it to the map of the Federal Republic of Germany, the following result was obtained (Fig. 3):



Fig. 3 The result of coloring with the Dsatur algorithm for the map of the Federal Republic of Germany

The result shows that for the same map, a more optimal coloring was obtained and four colors were used. By implementing an automated testing program that iterates through all the maps and returns the number of colors used for coloring each, the algorithms were verified for the remaining maps. It was found that in the case of greedy coloring, 6 countries were colored with 1 color, 16 with 2 colors, 73 with 3 colors, 96 with 4 colors, and 16 with 5 colors. The same check for the Dsatur algorithm showed that 6 countries were colored with 1 color, 16 with 2 colors, 87 with 3 colors, 92 with 4 colors, and 6 countries with 5 colors (Liechtenstein, Philippines, Puerto Rico, Uganda, Venezuela, Yemen). Although a better result was obtained, there are still countries that are colored with more than 4 colors (Fig. 4). Therefore, let us present another algorithm that will improve the obtained result.

To obtain a better coloring and solve the problem, the Backtracking algorithm [5] was used. The basic principle of the algorithm is as follows: *perform the coloring step by step, and if in one of the steps all four possible colors for the current vertex are exhausted, then go back one step and try to apply a different color to the previous vertex*. The main steps of the algorithm are as follows:

1. Order the vertices of the graph in decreasing order of vertex degree.

2. Select the current vertex and try to color it with the smallest possible color, trying the colors from the smallest to the largest.
3. If it is not possible to color the given vertex with any of the 4 colors, then go back to the previous vertex, change the color of that vertex and try again.

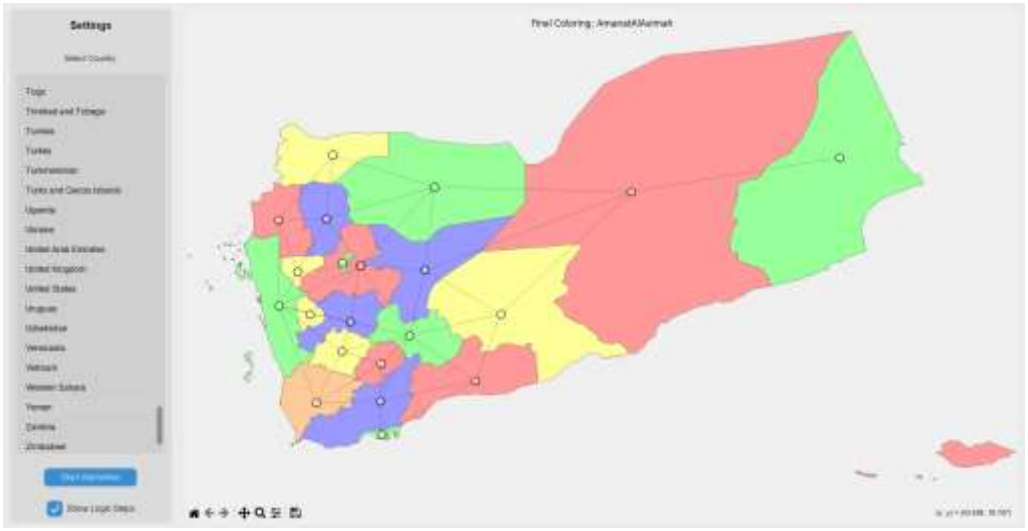

Fig. 4 The result of coloring with the Dsatur algorithm for the map of the Republic of Yemen

The concept of a recursive function was used in the software implementation. The backtrack function calls itself for the next vertex and returns "True" if the coloring was successful. Otherwise, the coloring of the given vertex is canceled and the vertex is colored with the next available color. This process continues until all vertices are colored or all possible search options are exhausted.

By implementing the coloring with this algorithm and applying the check, it turned out that 6 countries were colored with 1 color, 16 with 2 colors, 87 with 3 colors, 97 with 4 colors, and 1 country (Liechtenstein) with 5 colors.

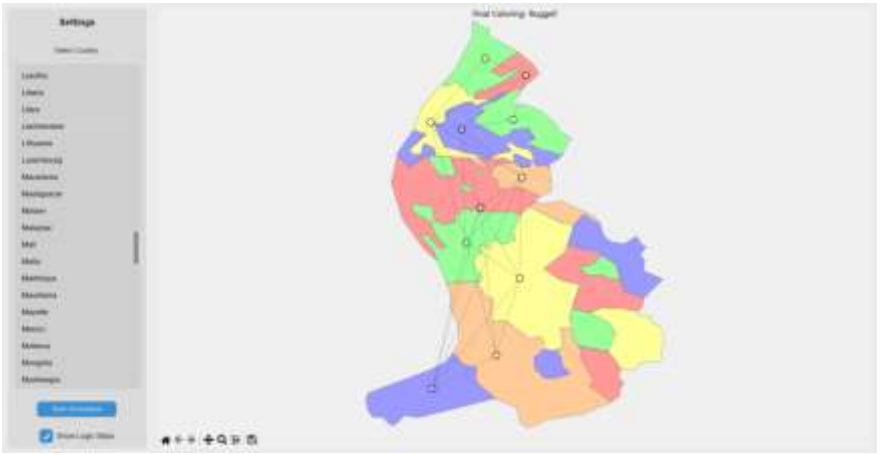

Fig. 5. The result of coloring with the Dsatur algorithm for the map of the Principality of Liechtenstein

Note that 5 colors were used when coloring the map of the Principality of Liechtenstein, since there are exclaves in the map, that is, the administrative-territorial unit is divided into several separate pieces and the geographical map contradicts the definition of a flat graph (an administrative-territorial unit is not a connected region). The figure below clearly shows that a number of polygons do not have a vertex, since they geographically belong to other administrative-territorial units (Fig. 5).

In this case, the map should be viewed not in terms of geographical units, but in terms of visible polygons (connected regions), where the polygons will be considered different administrative-territorial units. We will not address this in our work.

Research Results

By comparing the result of the backtracking algorithm with those previously obtained using Dsatur algorithm for the map of the Republic of Yemen – which had been colored with 5 colors, the following was got (Fig. 6):

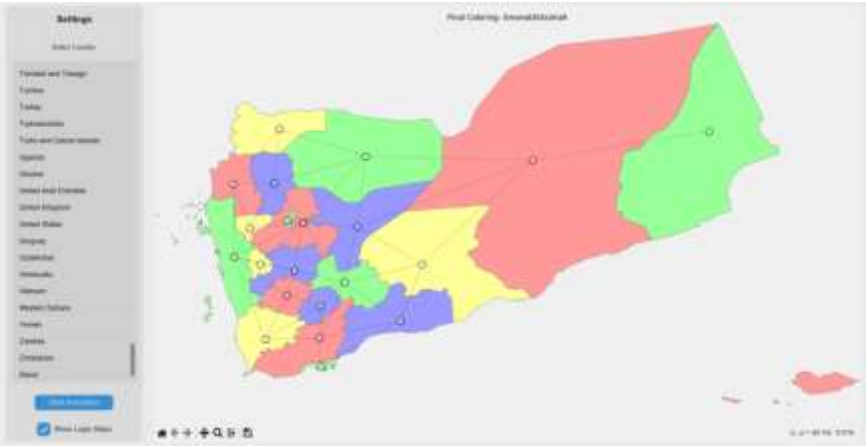

Fig. 6. Backtracking algorithm coloring result for the map of the Republic of Yemen

The more optimal result obtained shows that this algorithm can be used to verify the 4-color theorem. However, since the algorithm uses recursive functions, it can be very slow for countries with a large number of administrative units, such as the Russian Federation or the United States of America, which is not desirable. To circumvent this problem, a check has been added to the program and this algorithm is applied only for the 6 countries mentioned (for which the Dsatur algorithm uses five colors), and in other cases the faster Dsatur algorithm is used.

Conclusion

The study conducted during the development of the Four-Color Theorem visualization software tool showed that the errors in the geographical map data used do not allow for accurate verification of the proximity of administrative units and there is a need to use a safety buffer. The application of the Four-Color Theorem for maps divided by geographic units does not always give the desired result, since the enclaves and exclaves present in real geographic data contradict the construction of a planar graph. In such cases, the map is divided not geographically, but according to geometric polygons. The study and application of different coloring algorithms showed that the greedy coloring algorithm in the case of LDF ordering of vertices does not always give optimal results for real maps. The Dsatur algorithm finds a more optimal coloring, but in some cases more than 4 colors are used. The backtracking algorithm is the most optimal of the algorithms examined, but it is not advisable to use it in the entire program, since due to the use of recursive functions it works slowly with large amounts of input data. For this reason, a hybrid approach was used, using the two algorithms mentioned above depending on the country being colored. The created automatic testing tool allows to easily check the accuracy of coloring for a large number of maps. The developed software tool will help students researching the theorem to better understand the

essence of the theorem and visualize its application. The created software tool is available at the following link: https://github.com/nsarg4/four-color-theorem-vizualization

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ՉՈՐՍ ԳՈՒՅՆԵՐԻ ԹԵՈՐԵՄԻ ՊԱՏԿԵՐԱՎՈՐՄԱՆ ԿՐԹԱԿԱՆ ԾՐԱԳՐԻ ՄՇԱԿՈՒՄ

Ն.Ա. Սարգսյան
Երևանի պետական համալսարան

Չորս գույների թեորեմը գրաֆների տեսության ամենահետաքրքիր թեորեմներից մեկն է: Թեորեմի հետաքրքրաշարժությունը պայմանավորված է նաև նրանով, որ բացի խիստ մաթեմատիկական ձևակերպումից այն կարելի է նաև հեշտությամբ պատկերավոր ցուցադրել իրական քարտեզների վրա, ինչը թեորեմի բնույթը պարկերացնելն ավելի հեշտ է դարձնում: Այս հանգամանքը հատկապես կարևոր է կրթական ոլորտում, երբ պատկերավոր կամ սխեմատիկ բացատրությունը կարող է բավականաչափ հեշտացնել բարդ տեսական նյութի յուրացումը: Չնայած թեորեմն առաջ է քաշվել ավելի քան 170 տարի առաջ և այն խորապես ուսումնասիրված է, փոքրաթիվ են այնպիսի ծրագրային գործիքները, որոնք պատկերավոր կցուցադրեն թեորեմի կիրառումը իրական քարտեզների վրա: Առկա գործիքների մի մասը աշխատում է միայն կոնկրետ երկրի քարտեզի հետ, մյուսները հնարավորություն են տալիս ինքնուրույն գունավորել քարտեզը: Այս աշխատանքի նպատակն է նախագծել և մշակել ծրագրային մի գործիք, որը կցուցադրի ոչ միայն իրական քարտեզի ներկման արդյունքը, այլ նաև կիրառված ալգորիթմների բոլոր քայլերը՝ քարտեզից հարթ գրաֆի անցումը, գագաթների համարակալումը և գրաֆի գագաթային ճիշտ ներկումը: Աշխատանքում մշակված գործիքը հնարավորություն է տալիս կիրառել չորս գույների թեորեմը տարբեր երկրների

քարտեզների վրա, ինչպես նաև համեմատություն է կատարվում ներկման որոշ հայտնի ալգորիթմների միջև:

Բանալի բառեր. հարթ գրաֆներ, չորս գույների թեորեմ, գրաֆի գագաթային ներկում, քարտեզի ճիշտ ներկում, ագահ ալգորիթմ, Dsatur ալգորիթմ, backtracking ալգորիթմ

РАЗРАБОТКА ОБРАЗОВАТЕЛЬНОЙ ПРОГРАММЫ ДЛЯ ВИЗУАЛИЗАЦИИ ТЕОРЕМЫ О ЧЕТЫРЕХ КРАСКАХ

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Теорема о четырех красках — одна из самых интересных теорем в теории графов. Теорема также интересна тем, что, помимо строгой математической формулировки, её легко проиллюстрировать на реальных картах, что облегчает понимание её сути. Это особенно важно в образовательной сфере, где наглядное или схематическое объяснение может значительно облегчить понимание сложного теоретического материала. Хотя теорема была предложена более 170 лет назад и глубоко изучалась, существует немного программных инструментов, способных проиллюстрировать её применение к реальным картам. Некоторые из доступных инструментов работают только с картой конкретной страны, в то время как другие позволяют раскрасить карту самостоятельно. Цель данной работы — разработать программный инструмент, который будет отображать не только результат раскраски карты, но и все этапы применяемых алгоритмов: переход от карты к планарному графу, нумерацию вершин и правильную раскраску графа. Разработанный в данной работе инструмент позволяет применять теорему о четырех красках к картам разных стран, а также проводится сравнение некоторых известных алгоритмов раскраски.

Ключевые слова: Планарные графы, теорема о четырех красках, вершинная раскраска графа, правильная раскраска карты, жадный алгоритм, алгоритм Dsatur, поиск с возвратом.

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