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ON ONE PROBLEM OF PLANTS GROWTH DYNAMICS

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Abstract

In the current work, a mathematical model describing plant growth dynamics is considered. Constructive methods for the solution of derived non-linear ordinary differential equation are suggested. It has been shown that iteration method is more precise than quasi-linearized approach. At the end of the work numerical calculations for different starting values of water amount have been implemented. Results have been applied to the herbaceous and woody plants.

***Key words*: plant growth, water, intrinsic growth rate, degradation factor, nonlinearity, differential equation.**

Introduction

One of the most important problem of agronomy is the irrigation effect on crop growth. There are different mathematical models devoted to water supply in irrigated lands [1-5]. In this research [1] a mathematical model has been developed which allows to establish relationship between the water amount supplied to the plant and the plant growth. In accordance to this model, plant growth dynamics is described through the following nonlinear ordinary differential equation:

$$\frac{dm(t)}{dt} = \frac{\alpha(t)m(t)}{1+\beta m(t)} - \gamma m(t), t \in [0, T_0] , \quad (1)$$

where T_0 is the maximum period during which a plant grows, m(t)>0 is the amount of dry mass at t time, $\frac{dm(t)}{dt}$ corresponds to the growth rate of the plant due to the water inside the plant(mass/time), $\alpha(t)$ (1/time) is the intrinsic growth rate of the plant depending on the amount of supplied water ω(t) (mass):

$$\alpha(t) = k\omega(t), \tag{2}$$

here k ($mass^{-1} \times time^{-1}$) is the proportionality coefficient that considers the influence of water inside the plant on the growth of the crop. The water amount ω(t) inside the plant can be changed throughout the time due to transpiration and photosynthesis. Since the amount of dry mass can't grow permanently, the expression $[1+\beta m(t)]^{-1}$ is the limiting term of plant growth, β(1/mass) is the so called limiting factor. Finally, last term of equation (1) describes the rate of plant degradation, γ(1/time) is the degradation factor.

Initial condition is added to the equation (1).

$$m(0) = m_0 > 0. \tag{3}$$

Let's note that condition $m_0 > 0$ is required, otherwise if $m_0 = 0$ then initial problem (1), (3) will have only zero solution.

Conflict setting

In the current work the constructive methods for the solution of the initial problem (1), (3) have been recommended. Numerical calculations show that iterative method is more precise and close to the exact solution than quasi-linearized approach. For different initial values of dry mass numerical calculations for herbaceous and woody plants have been implemented. Computations have shown that the greater the supplied water amount inside plant is, the higher dry matter mass at the end of plant growth is resulted.

Solution of initial problems (1), (3).

As already mentioned above, the equation (1) represents nonlinear differential equation with variable coefficient $\alpha(t)$. The existence and uniqueness of solution of the problem (1), (3) immediately is followed by Cauchy theorem (see [6]). Below we suggest two constructive approaches for solving initial problems (1), (3).

a) **Iterative method**

It is easy to notice that Cauchy problem (1), (3) is equivalent to the following integral equation:

$$m(t) = m_0 + \int_0^t \frac{[(\alpha(\tau)-\gamma)m(\tau)-\gamma\beta m^2(\tau)]}{1+\beta m(\tau)} d\tau, \tag{4}$$

Let's consider the following Picard's sequence approximations:

$$m_{(n+1)}(t) = m_0 + \int_0^t \frac{[(\alpha(\tau)-\gamma)m_{(n)}(\tau)-\gamma\beta m_{(n)}^2(\tau)]}{1+\beta m_{(n)}(\tau)} d\tau, \tag{5}$$

n=0,1, 2, …

taking as a zero approximation:

$$m_{(0)} = m_{0.}, \tag{6}$$

b) **Quasi-linearization method**

The term $[1+\beta m]^{-1}$ standing in equation (1) is decomposed into Taylor series:

$$\frac{1}{1+\beta m} \approx 1 - \frac{\beta m}{1!} + \frac{\beta^2 m^2}{2!} - \cdots, \tag{7}$$

Ignoring β^2 and higher terms and taking into account (7), the equation (1) can be written as follows:

$$\frac{dm}{dt} = (\alpha(t) - \gamma)m(t) - \beta\alpha(t)m^2(t), \tag{8}$$

$$m(0) = m_0, \tag{9}$$

Let's note that the equation (8) represents well known Bernoulli's equation, which can be reduced to the first order linear differential equation with variable coefficient:

$$\frac{dz}{dt} + [\alpha(t) - \gamma]z(t) = \alpha(t)\beta, \tag{10}$$

with respect to unknown function:

$$z(t) = m^{-1}(t). \tag{11}$$

General solution of equation (10) has the following form:

$$z(t) = \exp[-\int_0^t (\alpha(\tau) - \gamma)\, d\tau]\left[\beta \int_0^t \alpha(\tau')\exp\,[\int_0^{\tau'} (\alpha(\tau'') - \gamma)d\tau''\right] d\tau' + c]. \tag{12}$$

Thus, solution of initial problem (8), (9) can be represented as follows:

$$m(t) = \frac{m_0}{\exp\left[-\int_0^t(\alpha(\tau)-\gamma)d\tau\right]+m_0\beta\exp\left[-\int_0^t(\alpha(\tau)-\gamma)d\tau\right][\int_0^t \alpha(\tau')\exp\,[\int_0^{\tau'}(\alpha(\tau'')-\gamma)d\tau'']d\tau'}. \tag{13}$$

Constant water supply

In this section we assume that water supply does not depend on time. For this purpose, we take an average value of function ω(t):

$$\alpha_0 = k < \omega >= \frac{k}{T}\int_0^{T_0} \omega(t)dt = k\omega_0. \tag{14}$$

Under assumption (14) the initial problem (1), (3) can be solved exactly. The equation (1) is presented in the following form:

$$\frac{(m(t) + r)dm}{m^2 + r\left(1 - \frac{\alpha_0}{\gamma}\right)m} = -\gamma dt, \tag{15}$$

where $r = 1/\beta$.

Integrating both parts of equation (15) and using initial condition (3) we get:

$$\ln\frac{m}{m_0}-\frac{\alpha_0}{\gamma}\ln\left|\frac{m+r(1-\frac{\alpha_0}{\gamma})}{m_0+r(1-\frac{\alpha_0}{\gamma})}\right|=(\alpha_0-\gamma)t, \qquad (16)$$

Let's take a note that equilibrium point of the equation (15) is equal to $\frac{\alpha_0-\gamma}{\beta\gamma}$. It should be mentioned that expression (16) implicitly gives exact solution to initial problem (1), (3) at $\alpha(t)=\alpha_0$.

Under assumption (14) the expression (13) is simplified and takes the form of:

$$m(t)=\frac{(\alpha_0-\gamma)m_0}{m_0\alpha_0\beta[1-e^{-(\alpha_0-\gamma)t}]+(\alpha_0-\gamma)e^{-(\alpha_0-\gamma)t}}. \qquad (17)$$

It is easy to see that

$$m(0)=m_0, \quad m(\infty)=\frac{\alpha_0-\gamma}{\beta\alpha_0}, \qquad (18)$$

The expression (17) represents approximate solution to the initial problem (1), (3), which has been resulted after quasi-linearization.

Research results

To make the developed approach more demonstrative, results of some numerical computations are introduced. The numerical calculations are implemented using package program written in the Python.

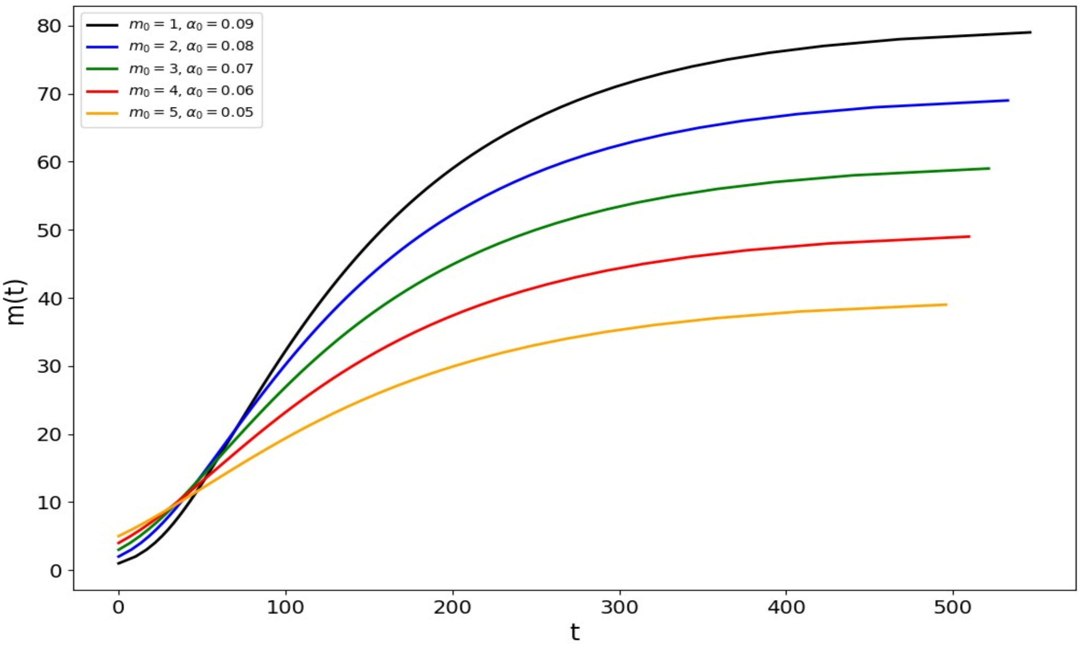

Fig. 1 The dependence of function m(t) on time

The parameters β = 0.1, k= 0.01, γ= 0.01 were taken from [1]. In herbaceous plants water makes approximately 70%-90% at the fresh weight and in woody plants it is more than 50% (see [7]). So, m_0= 3 (ω_0=7) for herbaceous plants and m_0= 5 (ω_0=5) for woody plants are taken as initial values.

In Fig. 1 simulations of dry mass growth as a function of time at different initial values of m_0 are plotted computed through the implicitly presented (16) formula, which gives exact solution to the problem.

For each initial amount of dry matter there exists inflection point t_0. Note that the smaller m_0 up to inflection point t_0 (t<t_0) is, the slower function m(t) grows up. If t>t_0, then function m(t) increases dramatically (the greater the supplied water amount, the higher the dry matter content at the end of the plant growth is).

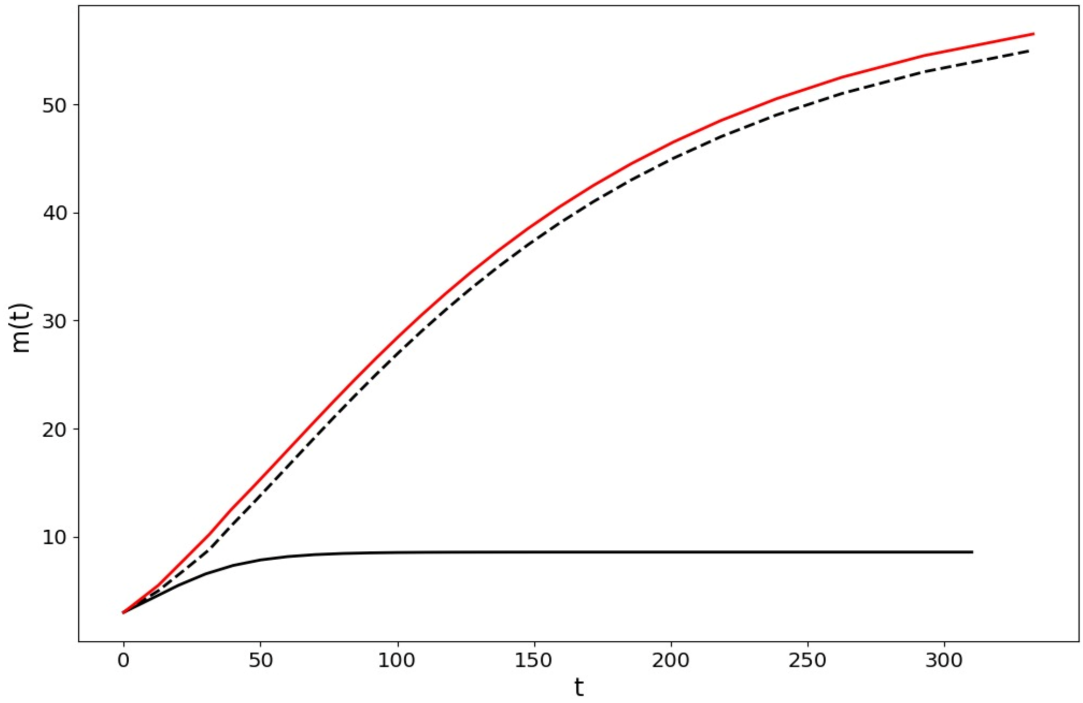

Fig. 2 The dependence of dry matter m(t) on the time for herbaceous plants

Fig. 2 shows dependence of dry matter m(t) (unit of mass) on the time for herbaceous plants. Dotted line corresponds to the exact solution introduced through (16) formula. Black solid line corresponds to the approximate solution represented by (17). With red line numerical results are plotted computed via iterations (5), (6).

Fig. 3 shows dependence of dry mass m(t) (unit of mass) on the time for woody plants. Dotted line corresponds to the exact solution derived through (16). Black solid line complies with the solution represented by (17). With red line results computed by iterations (5), (6) are plotted.

There is the following estimate between next and previous approximations [6]:

$$\left|m_{(n+1)} - m_n\right| \leq \frac{ML^nT^{n+1}}{(n+1)!},$$

where $M = max|f(m)|,\ m_0 \leq m \leq M_0$

$$f(m) = \frac{\alpha_0 m}{1+\beta m} - \gamma m,\ \ m_0 \leq m \leq \frac{\alpha_0 - \gamma}{\beta\gamma} (\equiv M_0)$$

$$T = min\left\{T_0, \frac{M_0 - m_0}{M}\right\}.$$

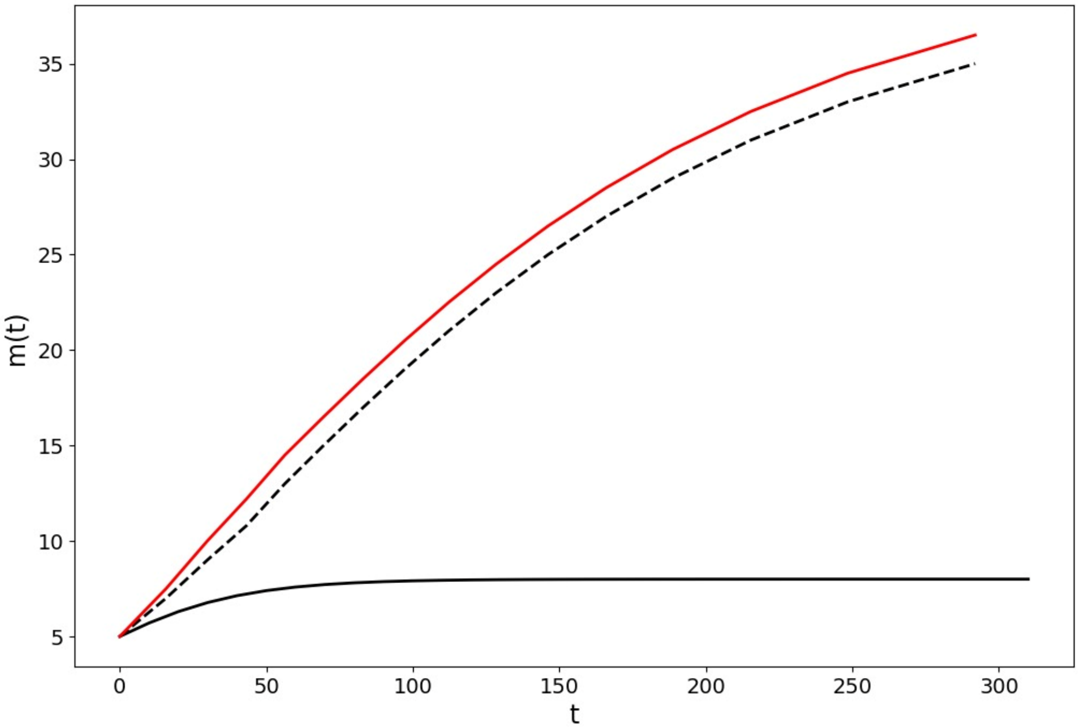

Fig. 3 The dependence of dry matter m(t) on the time for woody plants

It is easy to check that extremum point of function f(m) is equal to:

$$m_{crit} = \frac{\sqrt{\frac{\alpha_0}{\gamma}} - 1}{\beta},$$

Hence,

$$M = \frac{\alpha_0 + \gamma}{\beta} - \frac{2\sqrt{\alpha_0 \gamma}}{\beta} > 0,$$

L= $\frac{\alpha_0}{(1+\beta m_0)^2}$ is the Lipschitz coefficient .

The solution obtained by iterations is very close to the exact solution (only 2%-3% deviation).

The numerical calculations indicate that the solutions obtained by iterative and approximate (quasi-linearized) methods are close to the precise solution near zero. However, as the argument increases, the deviation of the approximate solution from the exact one becomes larger. Linearization of a non-linear problem can sometimes “distort” a “healthy” original non-linear problem both qualitatively and quantitatively. The case we are considering is a vivid example of a quantitative deviation from precise solution due to quasi-linearization.

Recommendations: For the comprehensive description of irrigation effect on the plants growth, a new model should be developed, which will consider not only the plants growth dynamics, but also water dynamics both in the root system and within the plant, including evaporation and photosynthesis under the impact of solar energy. We believe that it can be described through the non-linear differential equation system. To address the problem related to the development of the new model, another individual research work is supposed to initiate.

Conclusion

Considering the mathematical model describing plant growth dynamics, iterative and quasi-linearized approaches for solving the problem are suggested. It turns out, that iterative method is more precise than approximate approach. Numerical results show that the greater the supplied water amount within the plant is, the higher matter mass is observed at the end of plant growth.

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Եղիազարյան Գ.Մ., Խաչատրյան Ա.Խ., Դանիելյան Ռ.Ա.
Հայաստանի ազգային ագրարային համալսարան

Դիտարկվում է բույսերի աճի դինամիկան նկարագրող մի մաթեմատիկական մոդել: Առաջարկվում են առաջացող ոչ գծային դիֆերենցիալ հավասարման լուծման կոնստրուկտիվ մեթոդներ: Ցույց է տրվել, որ իտերացիոն մեթոդն ավելի ճշգրիտ է, քան քվազիգծային մոտեցումը: Կատարվել են թվային հաշվարկներ՝ ջրի քանակի տարբեր սկզբնական արժեքների համար: Արդյունքները կիրառվել են խոտային և բնափայտային բույսերի համար:

Բանալի բառեր. բույսի աճ, ջուր, աճի ներքին արագություն, քայքայման գործակից, ոչ գծայնություն, դիֆերենցիալ հավասարում:

ОБ ОДНОЙ ПРОБЛЕМЕ ДИНАМИКИ РОСТА РАСТЕНИЙ

Егиазарян Г.М., Хачатрян А.Х., Даниелян Р.А.
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Рассматривается математическая модель, описывающая динамику роста растений. Предложены конструктивные методы решения возникающего нелинейного обыкновенного дифференциального уравнения. Показано, что итерационный метод является более точным, чем квазилинеаризованный подход. Выполнены численные расчеты для различных начальных значений количества воды. Результаты применяют к травянистым и древесным растениям.

Ключевые слова: рост растений, вода, собственная скорость роста, коэффициент деградации, нелинейность, дифференциальное уравнение.

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