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PROPOSAL FOR KUSTAANHEIMO-STIEFEL TRANSFORMATION ON K¨AHLER MANIFOLD

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We propose the analog of Kustaanheimo-Steffel transformation on the K¨ahler manifold based on the symmetry properties of the latter. For this purpose we invent the alternative notion of the oscillator on K¨ahler manifold based on the existence of the so-called Fradkin tensor. Then we define the reduced coordinates as a Killing potentials of Kahler structure. We also presents the relations which provide the proposed oscillator model with the Lie algebra symmetries and find the analog of "monopole spin" in the reduced system.

Key words: Kustaanheimo-Steffel transformation, K¨ahler manifold, Killing potentials, Lie algebra symmetries, monopole spin.

Introduction

It is well-known that the radial Shrödinger equation for the $(p+1)-$dimensional Coulomb problem is transformed into that for $2p-$dimensional isotropic oscillator by the transformation

$$r = R^2, \tag{1}$$

where r and R denote the radial variable in Coulomb and oscillator problems, respectively. For p=1, 2, 4 one can establish the complete correspondence between these two systems.

For example, the two-dimensional Coulomb problem transforms into a two-dimensional oscillator by the so-called Levi-Chivita (or Bohlin) transformation [1]

$$z = w^2, \tag{2}$$

where z and w are the complex coordinates parameterizing configuration spaces of two-dimensional Coulomb and oscillator systems, respectively. This transformation maps the energy levels of the Coulomb problem into those of a circular oscillator given by the Hamiltonian

$$H = \frac{\pi\bar{\pi}}{2m} + 2m\,\omega^2 w\bar{w}, \tag{3}$$

while the Runge-Lenz vector transforms into the quadratics constant of motion of oscillator

$$I = \frac{1}{2m}\pi^2 - 2m\omega^2 w^2. \tag{4}$$

It is seen from (2) that quantum circular oscillator is in one-to-one correspondence with the two-dimensional quantum Coulomb system on the *two-sheet Riemann surface*. Thus, reducing the latter system by the Z_2 group action, corresponding to spatial reflections, one get that the two-dimensional Coulomb problem corresponds to even states of the oscillator one, while the odd states of the oscillator corresponds to the Coulomb problem, modified by the presence of magnetic flux that provides the system by $1/2$ spin [2].

The correspondence between three-dimensional Coulomb problems and a four-dimensional oscillator can be established by the use of the so-called Kustaanheimo-Stiefel transformation [3]

$$x^i = z\sigma^i \bar{z}. \tag{5}$$

Here x^i are the Euclidean coordinates in the three-dimensional Coulomb problem, z^α, $\alpha=1,2$ are the complex coordinates, describing the isotropic oscillator in C^2, and σ^i denote Pauli matrices.

To reduce the four-dimensional oscillator to the three-dimensional Coulomb system, we have to perform the Hamil-tonian reduction by the action of $U(1)$ group generated by the function

$$J_0 = i(z\pi - \overline{z\pi}) \tag{6}$$

given on the conangent bundle of C^2 equipped with the canonical symplectic structure

$$dz \wedge d\pi + d\bar{z} \wedge d\bar{\pi}\,. \tag{7}$$

The reduced symplectic structure takes the form

$$dx^i \wedge dp_i + s\frac{x^i}{r^3}\epsilon_{ijk} dx^i \wedge dx^i\,, r^2 = x^i x^i, \tag{8}$$

whereas energy level of the oscillator Hamiltonian

$$H = E\ ,\ H = \frac{\pi\bar{\pi}}{2m} + 2m\,\omega^2 z\bar{z} \tag{9}$$

reduces to the energy levelr of the following three-dimensional system

$$H_{red} = -2m\,\omega^2,\ H_{red} = \frac{p^2}{2m} + \frac{s^2}{2mr^2} - \frac{E}{r^2}. \tag{10}$$

Here s is the value of the generator J_0, and $p^i = \frac{i(\pi\sigma^i z - \bar{\pi}\sigma^i \bar{z}}{2(z\bar{z})}$.

So, the Coulomb problem corresponds to the $J_0=s=0$ level surface, whereas $J_0=s=0$ corresponds to the charge-dyon system proposed by Zwanziger [4] and re-derived by many authors [5-9]. The Runge-Lenz vector of three-dimensional system (10) corresponds to the constants of motion of a four-dimensional oscillator given by the expressions

$$I^i = \frac{1}{2m}\left(\pi\sigma^i\bar{\pi}\right) - 2m\omega^2\left(z\sigma^i\bar{z}\right). \tag{11}$$

The five-dimensional Coulomb problem and its generalization, specified by the presence of a five-dimensional $SU(2)$ monopole (corresponding to the BPST instanton), can be obtained from the eight-dimensional oscillator [10, 11, 12] defined on the two-dimensional quaternionic plane H^2 by the Hamiltonian reduction on $SU(2)$ group.

Conflict setting

The moral of the proposed derivations of two- and three- dimensional Coulomb problems is not only in the elegant explanation of hidden symmetry of the Coulomb problem given by Runge-Lenz like vectors, but also in the possibility to construct modified Coulomb systems, specified by the presence of monopoles and hidden symmetries. From this point of view, the restriction on the dimension of the initial (oscillator) configuration space looks rather artificial. It comes from the requirement to get the reduced (Coulomb) system with flat configuration space.

Indeed, the above-presented correspondences between Coulomb and oscillator systems is based on the Hopf maps $S^1/Z_2= S^1$, $S^3/U(1)=CP^1\cong S^2$, $S^7/SU(2)=HP^1\cong S^4$ (since S^1, S^2, S^4 parameterize the angular parts of the corresponding Coulomb problems). On the other hand, the sort of a monopole arising in the reduced system is uniquely defined by the structure of the reduction group.

In the present note we propose some preliminary construction, that seems to be a relevant generalization of the Kustaanheimo-Stiefel transformation. We show that Kähler manifolds are appropriate configuration spaces, on which an analog of the Kustaanheimo-Stiefel transformation can be formulated.

Research results

Consider Kähler manifold equipped with the metric

$$g_{a\bar{b}}dz^a d\bar{z}^b = \frac{\partial^2 K}{\partial z^a \partial \bar{z}^b} dz^a d\bar{z}^b, \tag{12}$$

and with the associated Poisson brackets

$$\{f,g\}_0 = i\frac{\partial f}{\partial z^a} g^{\bar{a}b}\frac{\partial g}{\partial z^b} - i\frac{\partial f}{\partial z^b} g^{\bar{a}b}\frac{\partial g}{\partial \bar{z}^a}, \quad g^{\bar{a}b}g_{b\bar{c}} = \delta^{\bar{a}}_{\bar{c}}. \tag{13}$$

The isometries of the Kähler structure are defined by the *holomorphic Hamiltonian vector fields*

$$V^\mu = V^{\mu a}(z)\frac{\partial}{\partial z^a} + \bar{V}^{\mu\bar{a}}(z)\frac{\partial}{\partial \bar{z}^a}, \quad [V^\mu, V^\nu] = 2C^{\mu\nu}_\lambda J^\lambda, \tag{14}$$

where

$$V^\mu = \{H^\mu, \}_0, \ \{H^\mu, H^\nu\}_0 = 2C^{\mu\nu}_\lambda J^\lambda, \quad \frac{\partial^2 H^\mu}{\partial z^a \partial z^b} - \Gamma^c_{ab}\frac{\partial H^\mu}{\partial z^c} = 0, \tag{15}$$

with the functions H^μ called Killing potentials. Free particle motion on this manifold is defined by the following canonical symplectic structure and Hamiltonian

$$dz^a \wedge d\pi_a + d\bar{z}^a \wedge d\bar{\pi}_a, \qquad D_0 = \frac{1}{2}g^{\bar{a}b}\pi_a\bar{\pi}_b. \tag{16}$$

This Hamiltonian defines covariant derivatives of the symmetric tensors

$$\{D_0, \} = \pi_a\nabla^a + \bar{\pi}_a\bar{\nabla}^a, \quad \bar{\nabla}^a = g^{\bar{a}b}\nabla_b, \nabla_b = \frac{\partial}{\partial z^a} + \Gamma^b_{ac}\pi_b\frac{\partial}{\partial \pi_c}. \tag{17}$$

The isometries (15) define the Noether constants of motion

$$J_\mu = \frac{1}{2}(V^{\mu a}\pi_a + \bar{V}^{\mu\bar{a}}\bar{\pi}_{\bar{a}}): \{D_0, J^\mu\} = 0, \{J^\mu, J^\nu\} = C^{\mu\nu}_\lambda J^\lambda. \tag{18}$$

Let introduce the following functions

$$2P^\mu \equiv \{D_0, H^\mu\} = i\left(V^{(\mu)a}\pi_a - \bar{V}^{(\mu)\bar{a}}\bar{\pi}_a\right), \quad 2I^\mu \equiv \{D_0, P^\mu\} + H^\mu = A^\mu + H^\mu \tag{19}$$

which have the following Poisson brackets

$$\{P^\mu, P^\nu\} = C^{\mu\nu}_\lambda J^\lambda, \quad \{P^\mu, J^\nu\} = C^{\mu\nu}_\lambda P^\lambda, \quad \{J^\mu, J^\nu\} = C^{\mu\nu}_\lambda J^\lambda, \tag{20}$$
$$\{I^\mu, I^\nu\} = C^{\mu\nu}_\lambda J^\lambda + \frac{1}{4}\{A^\mu, A^\nu\}.$$

In the case when the following equation holds,

$$\{A^\mu, A^\nu\} \equiv \{H^{[\mu}, \nabla^4 H^{\nu}\} = 0, \tag{21}$$

the above functions form Lie algebra, which looks similar to the algebra of oscillator. The functions I_μ define to the Hamiltonian of an oscillator and its quadratic constants of motion. So, the equation (18) and (19) define the appropriate generalization of oscillator algebra.

In principle, one can consider a weaker condition on the system under consideration, requiring the existence of isometry H_0 obeying the conditions

$$\{H_0, J_\nu\} = 0, \ \{A_0, A_\nu\} = 0. \tag{22}$$

In that case, I_μ can also be considered as appropriate generalization of the oscillator Hamiltonian and Demkov's tensor.

On the other hand, taking into account (15), one can transform equations (21) containing the fourth-order derivatives of H_μ into equivalent ones containing only the first- and the second-order derivatives of H_μ.

It follows from (22) that J0commutes with all the functions H^μ, J^μ, P^μ, I^μ. Thus, reducing the system by the action of J_0, we can get the analog of of the Coulomb problem with I^μ defining the Hamiltonian and Runge-Lenz vector.

For this purpose we define the dual "momenta" P_μ conjugated with H^μ and the "dual metric" $g^{\mu\nu}$ (with the inverse metric $g^{\mu\nu}$)

$$P_\mu = g_{\mu\nu} P^\nu, \quad g^{\mu\nu}(H) \equiv \frac{\partial H^{\{\mu}}{\partial z^a} g^{\bar{a}b} \frac{\partial H^{\nu\}}}{\partial z^b} \tag{23}$$

It follows from (20) that the Poisson brackets of the functions P_μ, H_μ, and J_μ are of the form

$$\begin{cases} \{J^\mu, J^\nu\} = C^{\mu\nu}_\lambda J^\lambda, \quad \{H^\mu, J^\nu\} = C^{\mu\nu}_\lambda H^\lambda, \ \{P_\mu, J^\nu\} = C^{\mu\nu}_\lambda P_\lambda \\ \{H^\mu, H^\nu\} = 0, \ \{H^\mu, P_\nu\} = \delta^\mu_\nu, \ \{P_\mu, P_\nu\} = g_{\mu\alpha} g_{\nu\beta} C^{\alpha\beta}_\lambda \left(J^\lambda - C^{\lambda\beta}_\alpha H^\alpha P_\beta\right). \end{cases} \tag{24}$$

Hence, choosing the functionally independent generators H^i and conjugated momenta P_i, we get Poisson brackets describing particle in an external magnetic field $F_{\mu\nu} = g_{\mu\alpha} g_{\nu\beta} C^{\alpha\beta}_\lambda S^\lambda$, where S^λ is a "monopole spin"

$$J^\mu = C^{\mu\beta}_\alpha H^\alpha P_\beta + S^\mu. \tag{25}$$

It is clear, that on the flat spaces this construction coincides with formulae arising in the Kustaanheimo-Stiefel transformation which were given in the Introduction. Thus, it can be viewed as generalization of the Kustaanheimo-Stiefel transformation to Kähler manifolds.

Conclusion

In this paper we proposed the analog analog of harmonic oscillator on the generic Kahler manifold, and the generalization of Kustaanheimo-Stieffel transformation transforming this system to the Coulomb-like one. Our construction of the "generalaized oscillator" is based on the symmetry algebra of configuration space, and is different from the so-called "Kahler oscillator" construction [12, 13] which was inspired by the supersymmetry arguments and by the concrete model of oscillator system on complex projective spaces [14, 15]. For sure, our construction is quite preliminary and should be studied in more details.

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ԿՈՒՍՏԱԱՆԵՅՄՈ-ՇՏԵՖԼԻ ՁԵՎԱՓՈԽՈՒԹՅՈՒՆՆԵՐԻ ԿԻՐԱՌՈՒՄԸ ԿԷԼԵՐԻ ԲԱԶՄԱՁԵՎՈՒԹՅՈՒՆՆԵՐԻ ՎՐԱ

Գ.Մ. Բաղունց, Դ.Գ. Առստամյան, Մ.Գ. Պետրոսյան
Շուշիի տեխնոլոգիական համալսարան

Ներկայացված է Կուստաանեյմո-Շտեֆելի ձևափոխությունների անալոգ ձևը Կէլերի բազմության վրա, հիմնավորված վերջինիս համաչափային հատկություններով. Դրա համար մենք առաջարկում ենք օսցիլյատորի այլընտրանքային սահմանումը Կէլերի բազմության վրա, հիմնված այսպես կոչված Ֆրադկինի տենզորի գոյությամբ, իսկ հետո սահմանվում է ռեդուկցված կոորդինատները որպես կուլերյան կառուցվածքի Կիլլինգի պոտենցիալներ Մենք նաև բերում ենք առնչություններ, որոնք ապահովում են օսցիլյատորի առաջարկված մոդելը Լիի հանրահաշվի համաչափություններում և թույլ են տալիս գտնել “մոնոպոլային սպինի” անալոգը ռեդուկցված համակարգում:

Բանալի բառեր. Կուստաանեյմո-Շտեֆելի ձևափոխություններ, Կէլերի բազմություն, Կիլլինգի պոտենցիալներ, Լիի հանրահաշիվ համաչափություն, “մոնոպոլային սպինի”:

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ПРИМЕНЕНИЯ ПРЕОБАРОЗОВАНИЙ КУСТААНХЕЙМО-ШТЕФФЕЛЯ НА МНОГОБРАЗИИ КЭЛЕРА

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Предложен аналог преобразования Кустаанхеймо-Штеффеля на Келеровом многообразии многообразии, основанный на симметричных свойствах последнего. Для этого мы предлагаем альтернативное определение осциллятора на Келеровом многообразии, основанное на существовании так называемого тензора Фрадкина, а затем определяем редуцированные координаты как потенциалы Киллинга кулеровой структуры. Мы также приводим соотношения, которые обеспечивают предлагаемую модель осциллятора симметриями алгебры Лии находят аналог "монопольного спина" в редуцированной системе.

***Ключевые слова*:** Преобразования Кустаанхеймо-Штеффеля, Келеровом многообразие, потенциалы Киллинга, симметриями алгебры Ли, "монопольного спина".

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